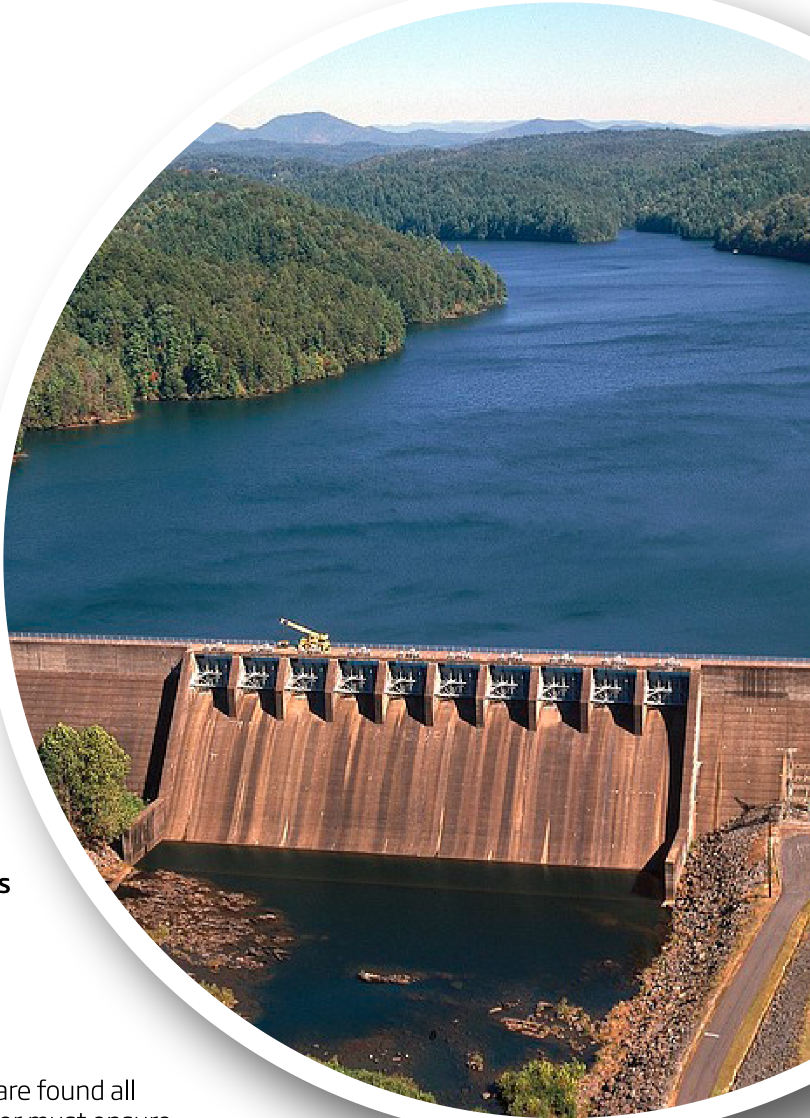
Force on a dam wall



Topic areas

Mechanical engineering:

- Forces
- Reaction
- Pressure
- Moments

Mathematics:

- Algebra
- Areas and volumes
- Arithmetic
- Integration

Prerequisites

None for the main activity, however it may be useful to look at the resource **'Centre of gravity, reactions and stability'** and **'Friction and sliding'** for the stretch and challenge activity.

Problem statement

Dams are important engineering constructions and are found all around the world. When designing a dam, an engineer must ensure that its construction will be strong enough to hold back the weight of water that will accumulate behind it. How can an engineer determine the force exerted by the water?

ROYAL ACADEMY OF ENGINEERING

MOTOROLA SOLUTIONS FOUNDATION

Background – pressure and force

Pressure is the total force acting on a given area. It has units of Nm^{-2}, or uses the derived unit of the pascal, Pa. The pressure is a static fluid at a depth h below the surface is given by the expression

$$p = \rho g h,$$

where ρ is the density of the fluid, g is the acceleration due to gravity and h is the depth below the surface. As pressure is a measure of the total force acting on a given area, and in the above, the force derives from the weight of the fluid, it makes sense that the pressure increases with depth as there's more mass of water above the point of measurement.

Look at the diagram in **Figure 1** which shows a schematic of a body of water that is being retained by a dam wall.

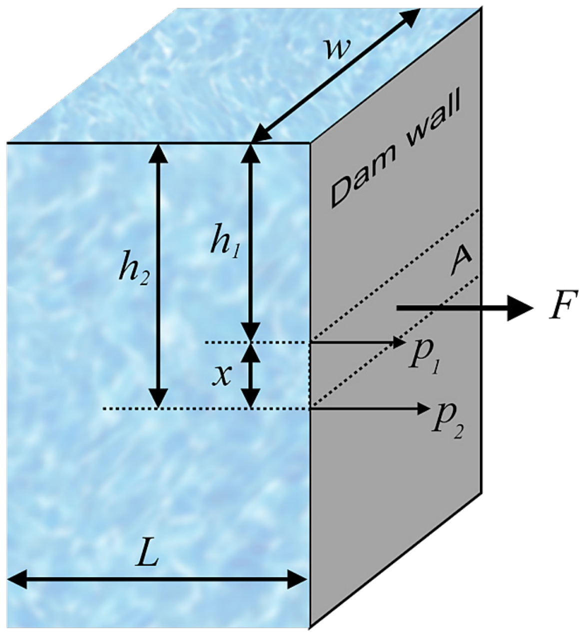


Figure 1
A dam wall

As $h_2 > h_1$, it is obvious that $p_2 > p_1$. This means that the force acting on a small area near the top of the area is smaller than the force acting on a small area near the bottom. However, because the pressure changes linearly with depth, which means the force acting on each small area also changes linearly with depth, the average pressure over A, given by $\bar{p} = \frac{1}{2}(p_1 + p_2)$, can be used to calculate the force on the area.

Activity 1 – Total force on a dam wall

1. Discuss the pressures and how you would calculate the total force, F, acting on the highlighted area A of the dam wall.
2. Does the force depend on all the items shown on the diagram?

Activity 2 – Total force on a dam wall

The resource dam-1, shown in **Figure 2**, considers a dam of width 5 m, and a height that can be varied by the user in 2 m increments by moving the slider on the right. In calculations it is assumed that ρ = 1000 kgm^{-3} and g = 10 ms^{-2}. Pressure values and individual section forces can be shown by clicking/tapping on the dam at the desired depth location.

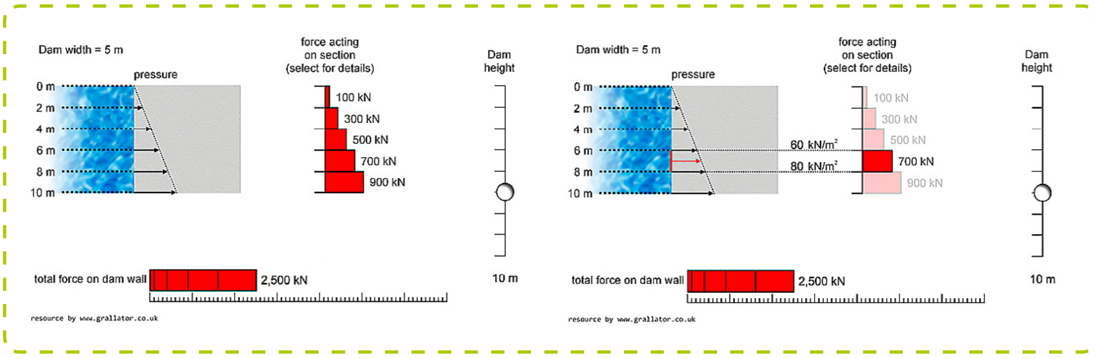


Figure 2
Screen shots of resource

Experiment with different dam heights. Look at the individual forces on each 2 m section of wall and how it changes with depth. Also look at how the total force on the wall changes with dam height.

The force on each 2 m section of wall increases linearly with depth. However, as each section adds more and more to the total force, the total force acting on the dam does not increase linearly with dam height. (This could be plotted as an extra activity.)

The force on each 2 m section of wall increases linearly with depth. The first few values are (in kN)

100, 300, 500,...

These values form an arithmetic series of the form $F_n = a + (n-1)d$, where a = 100 and d = 200, with $n = 1, 2, 3, \ldots$, i.e. $F_n = 100 + 200\,(n-1)$. Note, the maximum value of n is determined by the height of the dam, so, for example, a 10 m high dam will have a maximum value of n = 5 (as each n-section is 2m high). Generalising, the height of the dam, H, is given by twice the maximum value of n, or, conversely, the maximum value of n is given by half the dam height, H.

It can be shown that the sum to N terms of an arithmetic progression is given by

$$S_N = \frac{1}{2}N\left(2a + \left(N-1\right)d\right).$$

As this gives the sum of all the forces acting on each 2m section, this expression gives the total force acting on the dam.

1. Use the expression above to write an expression for the total force on the dam wall of height H. *Hint: remember H = 2N, where N is the maximum value of n.*
2. What type of function describes the relationship between the total height of the dam and the total force of water acting on it?

Background – Finding the total force using integration

The total force on a dam wall was found by summing up the forces on a number of elements. In the activity above, the forces on each element changed linearly so it did not matter how many elements the total height was broken down into. However, this is not always the case, for example when moments are calculated, and in such calculations a different method is required: integration (as will be apparent in Activity 4).

Reducing the size of each height element to an infinitesimally small height δx at a depth x below the surface and summing over all such elements gives a method of calculating the total force by integration.

The pressure at a depth x is given by $p = \rho g x$.

The force acting on a wall section of height δx and width w is $\delta F = pA = \rho g x \cdot w \cdot \delta x$

Note that δx is so small that pressure differences between the top and bottom are negligible.

In the limit $\delta x \to 0$, the total force F is given by the integral $F = \int_0^H \rho g x w \, dx$

Activity 3 – Evaluate the integrate

Evaluate the integral and show that when $\rho = 1000\ \text{kgm}^{-3}$ and $g = 10\ \text{ms}^{-2}$, the total force varies with dam height H as discovered in Activity 2.

Background – the centre of pressure

The centre of gravity of an object is the average position of the weight of an object, and the object's motion can be completely described by the translation of the centre of gravity, and by rotations about the centre of gravity. A similar concept can be used for the pressure acting on an object due to a fluid. This is called the centre of pressure. The centre of pressure gives a force acting a single location that is equivalent in magnitude and turning moment to all the pressure forces combined, see **Figure 3**.

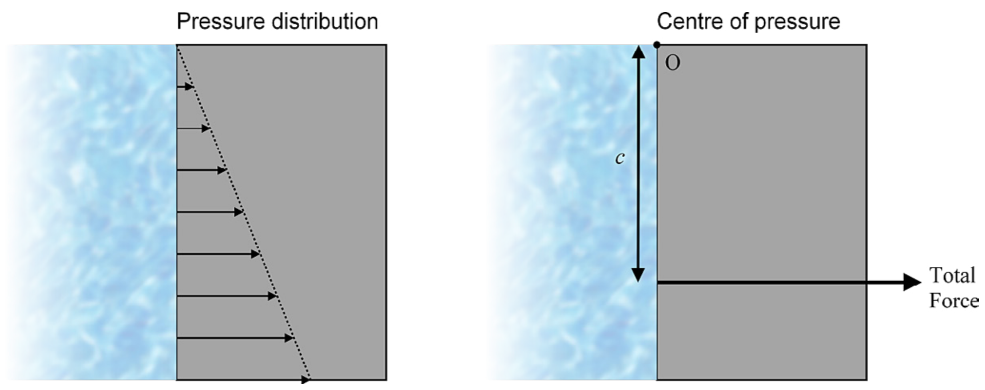


Figure 3
Centre of pressure

The magnitude of the total equivalent force has been found in Activities 2 and 3. To find the depth below the surface, c, at which this acts to give an equivalent turning moment requires taking moments of all the forces along the wall of the dam. It is convenient to take the moments about the position O on the surface as this is also the reference location for depth measurement.

Activity 4 – Calculating the centre of pressure

1. In Activity 3 you found the force acting on a wall section of height δx and width w, a distance x below O is $\delta F = pA = \rho g x \cdot w \cdot \delta x$. What is the moment of this force about O?
2. Taking the limit $\delta x \to 0$, write down an integration expression for the total moment about O.
3. The moment of the total force F about O is given by Fc (see right hand shape in **Figure 4**). Use this and the result from 2 to calculate c in terms of the dam height H.

Stretch and challenge activity

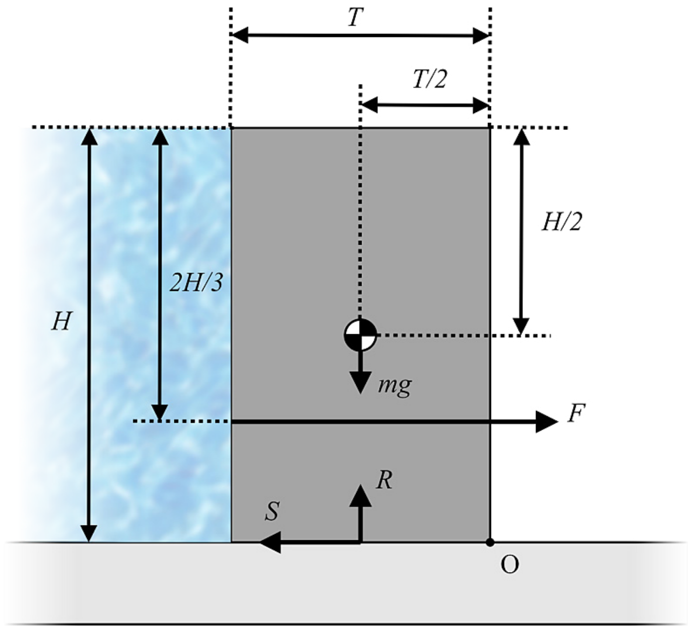


Figure 4
A gravity dam

A gravity dam is one which uses just its own weight to hold back the water behind it. For the dam to be stable it must not slide or tip over.

The diagram in **Figure 4** shows a side view of a gravity dam constructed using a rectangular cuboid block of concrete of height H = 6 m, and thickness T. The width of the dam is w = 5 m (not shown on the diagram). The block has a density of 2,400 kgm^{-3} and sits on an impermeable surface. The coefficient of friction between the block and the surface is μ = 0.8.

The weight of the block is mg newtons, acting through the centre of the block. The hydrostatic pressure is F newtons acting through the centre of pressure as shown. Assuming water has a density of 1,000 kgm^{-3},

1. What do the forces R and S represent?
2. Write a relationship between R and S.
3. Write an expression for the mass of the block in terms of its thickness, T.

   Hint: density, ρ, is the mass, m, per unit volume, V, i.e. $\rho = \dfrac{m}{V} \Rightarrow m = \rho V$.
4. What is the minimum value of T to prevent the water pressure from making the dam block slide? *Hint: the limiting friction value is given by the product of the coefficient of friction and the normal reaction,* $S = \mu R$.
5. What is the minimum value of T to prevent the water pressure from making the dam block tip about O? *Hint look at moments from centre of pressure – note the location of O is different in* ***Figure 4***.
6. Which value of T should you use in the design?

Notes and solutions

Background - pressure and force

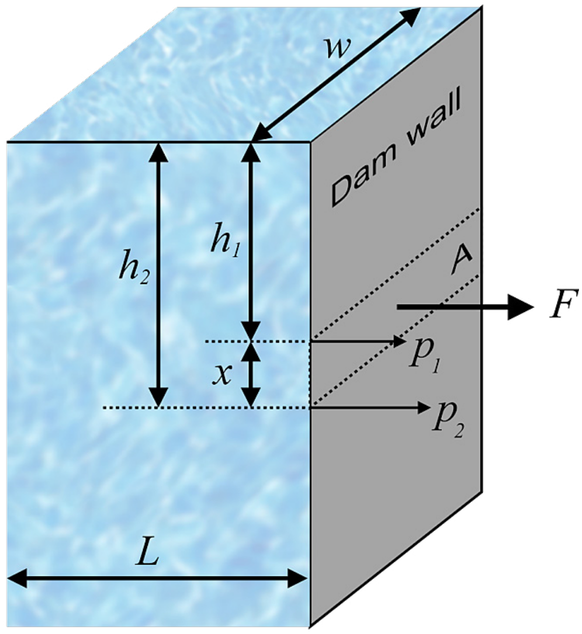


Figure 1 (repeated)
A dam wall

1. The pressures p_1 and p_2 are given by $p_1 = \rho g h_1$ and $p_2 = \rho g h_2$ respectively. These pressures exist at all points in the fluid at the same depths. Specifically the pressure p_1 is exerted at all points along the dotted line at the top of the area A, and similarly for p_2 on the dotted line at the bottom of area A.

   As $h_2 > h_1$, it is obvious that $p_2 > p_1$. This means that the force acting on a small area near the top of the area is smaller than the force acting on a small area near the bottom. However, because the pressure changes linearly with depth, the average pressure over A, given by $\bar{p} = \frac{1}{2}(p_1 + p_2)$, can be used to calculate the force on the area, as shown below

$$F = \frac{1}{2}(p_1 + p_2)A = \frac{1}{2}\rho g(h_1 + h_2)A = \frac{1}{2}\rho g(h_1 + h_2)wx$$

   This is just the pressure at the mid-point depth of the area A, multiplied by the area.

2. Note that in the above the force acting on the area A does not depend on the length of the body of water behind the dam, L. The only important dimensions are the width of the dam and the depth of the water.

Activity 2 - Total force on a dam wall

1. The total force, F, is given by the sum to N terms of the arithmetic series:

$$F = S_n = \frac{1}{2}N\left(2a + \left(N - 1\right)d\right)$$

Substituting a = 100 and d = 200,

$$\begin{aligned} F &= \frac{1}{2}N\left(200 + 200\left(N - 1\right)\right) \\ &= \frac{1}{2}N \cdot 200\left(1 + N - 1\right) \\ &= 100N^2 \end{aligned}$$

Each section is 2 m high, therefore N sections gives a height $H = 2N$ metres. Substituting into the above

$$\begin{aligned} F &= 100N^2 \\ &= 100\left(\frac{H}{2}\right)^2 \\ &= 25H^2 \quad (\text{kN}) \end{aligned}$$

2. This result shows the total force of water acting on a dam depends on the square of the height of the dam, i.e. it is quadratically dependent on H. You can explore this further by using the resource **dam-1** and looking at the total force at 1, 2, 4, 8 and 16 m heights.

Activity 3 - Finding the total force using integration

$F = \int_0^H \rho g x w \, dx = \rho g w \int_0^H x \, dx$, as ρ, g and w are independent of x

$$F = \rho g w \int_0^H x \, dx$$
$$= \rho g w \left[\frac{x^2}{2} \right]_0^H$$
$$= \ldots$$
$$= \frac{1}{2} \rho g w H^2$$

i.e. the total force varies quadratically with the dam height, as found in Activity 2.

Substituting $\rho = 1000 \text{ kgm}^{-3}$, $g = 10 \text{ ms}^{-2}$ and $w = 5$ m, gives

$$F = \frac{1}{2} \rho g w H^2$$
$$= \frac{1}{2} \cdot 1000 \cdot 10 \cdot 5 H^2$$
$$= 25{,}000 H^2 \quad (\text{N})$$

Note, the above is given in newtons as all the values are SI. Converting the kN, by dividing by 1,000, gives the total force as $F = 25H^2$ (kN), as found in Activity 3.

Activity 4 - Calculating the centre of pressure

1 The force acting on a wall section of height δx and width w, a distance x below O is $\delta F = pA = \rho g x \cdot w \cdot \delta x$. The moment of this force about O is given by the force multiplied by the perpendicular distance at which it acts from O, i.e.

$$\delta M = \delta F \cdot x = \rho g x \cdot w \cdot \delta x \cdot x = \rho g w x^2 \cdot \delta x$$

2 In the limit $\delta x \to 0$, the total moment about O, M, is given by the integral $M = \int_0^H \rho g w x^2 \, dx$.

$M = \int_0^H \rho g w x^2 \, dx = \rho g w \int_0^H x^2 \, dx$, as ρ, g and w are independent of x.

$$\begin{aligned} M &= \rho g w \int_0^H x^2 \, dx \\ &= \rho g w \left[\frac{x^3}{3}\right]_0^H \quad , \\ &= \ldots \\ &= \frac{1}{3}\rho g w H^3 \end{aligned}$$

The total moment varies as the cube of the dam height.

3 Moment of F about O is Fc. For c at the centre of pressure this must be equivalent to M, calculated above. F has been found in Activity 3 to be $F = \frac{1}{2}\rho g w H^2$. Using these:

$$\begin{aligned} Fc &= M \\ \frac{1}{2}\rho g w H^2 \times c &= \frac{1}{3}\rho g w H^3 \\ c &= \frac{2\rho g w H^3}{3\rho g w H^2} \\ c &= \frac{2}{3}H \end{aligned}$$

i.e. the centre of pressure is two-thirds of the height of the dam below the surface of the water.

Earlier in this Activity you saw that each small contribution to the total moment varies with the square of the depth, which means that forces near the bottom of the dam contribute more to the total moment than forces near the top. This skews the average moment position to be nearer the bottom than the top and explains why the average position is two-thirds of the height of the dam below the surface of the water.

Stretch and challenge activity

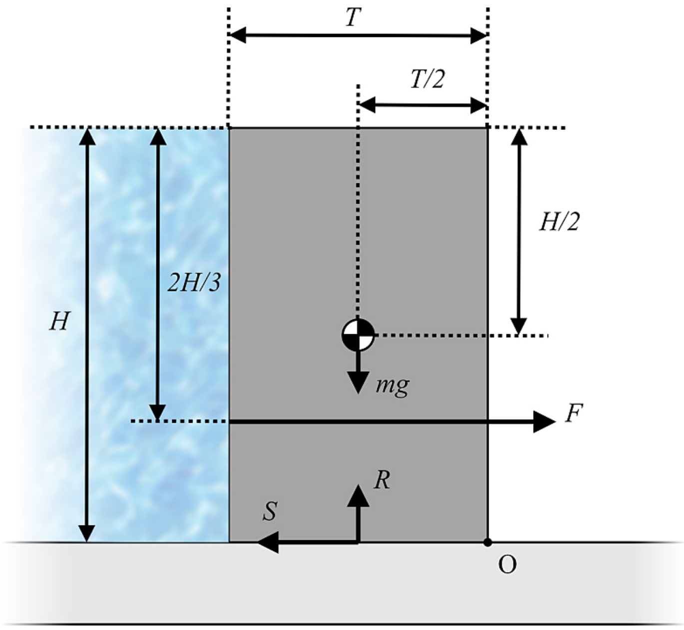


Figure 4 (repeated)
A gravity dam

Recall H = 6 m, width of the dam is w = 5 m. The block has a density of 2,400 kgm^{-3}. The coefficient of friction between the block and the surface is μ = 0.8.

1. R is the normal reaction from block's contact with the impermeable surface. S is the frictional force resisting sliding.

2. $S \leq \mu R$

3. The volume of the block is $V = HwT = 6{\times}5{\times}T = 30T$ (m^3).

   The mass of the block is given by the volume multiplied by the density,

   $m = \rho V = 2400 \times 30T = 72{,}000T$ (kg)

4 The normal reaction, R, is given by $R = mg = 72{,}000Tg$ in newtons. As $S \le \mu R$, the limiting friction force is given as $S = \mu R = 0.8 \times 72{,}000Tg = 57{,}600Tg$ in newtons.

The hydrostatic force on the dam from the water is given by $F = \frac{1}{2}\rho g w H^2$. Substituting values,

$$F = \frac{1}{2} \times 1{,}000 \times g \times 5 \times 6^2 = 90{,}000g \text{ in newtons.}$$

The frictional force must be at least this large to prevent sliding, so that $S = F$, i.e,

$$\begin{aligned} 57{,}600Tg &= 90{,}000g \\ T &= \frac{90{,}000g}{57{,}600g} \\ &= 1.5625 \quad (m) \end{aligned}$$

If T is less than 1.5625 m wide the block will not be heavy enough for friction to prevent it sliding against the pressure of the water.

5 The clockwise turning moment about O is due to the hydrostatic force acting through the centre of pressure. This moment is given by

$$M_c = F \times \frac{H}{3} = 90{,}000g \times \frac{6}{3} = 180{,}000g \text{ in Nm.}$$

The anticlockwise turning moment about O is due to the weight acting through the centre of gravity. This moment is given by

$$M_{ac} = mg \times \frac{T}{2} = 72{,}000Tg \times \frac{T}{2} = 36{,}000gT^2 \text{ in Nm.}$$

The anti-clockwise moment from the weight must be at least as large as the clockwise moment from the hydrostatic pressure, i.e. $M_{ac} = M_c$,

$$\begin{aligned} 36{,}000gT^2 &= 180{,}000g \\ T^2 &= \frac{180{,}000g}{36{,}000g} = 5 \\ T &= \sqrt{5} = 2.2361 \quad (\text{m} \quad 4 \text{ d.p.}) \end{aligned}$$

6 As the required thickness to prevent tipping is larger than the required thickness to prevent sliding, a dam thickness of at least 2.2361 m (4 d.p.) is required. A good engineer will add in a safety margin and make it wider than this!

ROYAL ACADEMY OF ENGINEERING

Royal Academy of Engineering

As the UK's national academy for engineering, we bring together the most successful and talented engineers for a shared purpose: to advance and promote excellence in engineering.

We have four strategic challenges:

Make the UK the leading nation for engineering innovation

Supporting the development of successful engineering innovation and businesses in the UK in order to create wealth, employment and benefit for the nation.

Address the engineering skills crisis

Meeting the UK's needs by inspiring a generation of young people from all backgrounds and equipping them with the high quality skills they need for a rewarding career in engineering.

Position engineering at the heart of society

Improving public awareness and recognition of the crucial role of engineers everywhere.

Lead the profession

Harnessing the expertise, energy and capacity of the profession to provide strategic direction for engineering and collaborate on solutions to engineering grand challenges.

Royal Academy of Engineering
Prince Philip House, 3 Carlton House Terrace, London SW1Y 5DG

Tel: +44 (0)20 7766 0600
www.raeng.org.uk

MOTOROLA SOLUTIONS FOUNDATION