Fireboat projectiles - SUVAT equations in two dimensions

Mechanical engineering

- Displacement
- Speed
- Velocity
- Acceleration
- SUVAT equations
- Resolving a vector into two perpendicular components
- Projectile motion
- Fluid flow rate

Mathematics

- Algebra
- Trigonometry
- Trigonometric identities
- Quadratic

Prerequisites

It is recommended that the resource **Motion of a tube train - SUVAT** is covered before using this resource. It would also be useful to have an awareness of resolving the horizontal and vertical components of velocity.

Problem statement

Fireboats are used to fight fires in coastal areas to protect, for example, boats in marinas, bridges over rivers and waterside property. They are also used offshore to, for example, fight fires on oil rigs.

In order to be effective, they must be able to pump enough water at sufficient speed to reach a fire from a safe distance, and also be able to aim the water stream high enough to reach fires that may be significantly above the water level.

How can the path of the water stream be described mathematically and how can this be used to predict height and reach?

ROYAL ACADEMY OF ENGINEERING

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Activity 1 – Discussion

Figure 1 shows a simplified version of one of the water streams emerging from a nozzle on the boat in the image on the front page.

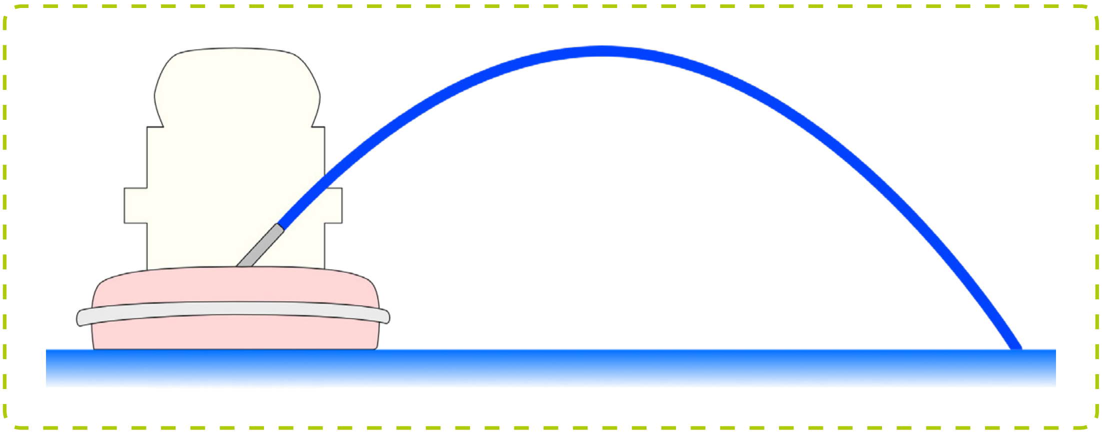

Figure 1 A simplified image of a fireboat

1. In terms of forces and motion, what is happening to the water as it leaves the nozzle?
2. What kind of function forms a plot with a shape that is similar to the path of water? *Hint: turn the paper upside down – it may help!*

Background – projectile motion

Projectile motion arises when two components of motion are combined for a single object:

- A horizontal motion in the x-direction, assumed to be at constant speed, i.e. no acceleration, so that $a_x = 0$.
- A vertical motion in the y-direction, assumed to be under gravity, so that $a_y = -g$ (= −10 ms^{-2} for these examples).

The horizontal and vertical motions are independent of each other. The horizontal and vertical displacements are given as a function of time from the SUVAT equation $s = ut + \frac{1}{2}at^2$ as

Horizontal: $s_x = u_x t + \frac{1}{2} a_x t^2 = u_x t$

Vertical: $s_y = u_y t + \frac{1}{2} a_y t^2 = u_y t - \frac{1}{2} g t^2$

Where u_x and u_y are the initial velocities in the horizontal and vertical directions respectively, usually taken as being positive for velocity to the right or up, and negative for velocity to the left or down. It is for this reason that g appears as a negative value in the second equation.

The path of water from a nozzle can be modelled by assuming that the stream is composed of a large number of small 'packets' of water, each of which obeys projectile motion. The equations of motion given above under this assumption lead to the water stream having the shape discussed in Activity 1.

1. What other assumption has been made?

Activity 2 - Experimenting with projectile motion

The resource **SUVATprojectile1.html** allows you to experiment with the projectile motion of a particle. In the initial configuration with (u_x, u_y) highlighted, as shown in **Figure 2**, you can independently set the horizontal and vertical components of the initial velocities and plot the projectile path followed. Three views are presented: a side view showing x and y motion, a top view that shows only the x-component of motion, and a front view that shows only the y-component of motion.

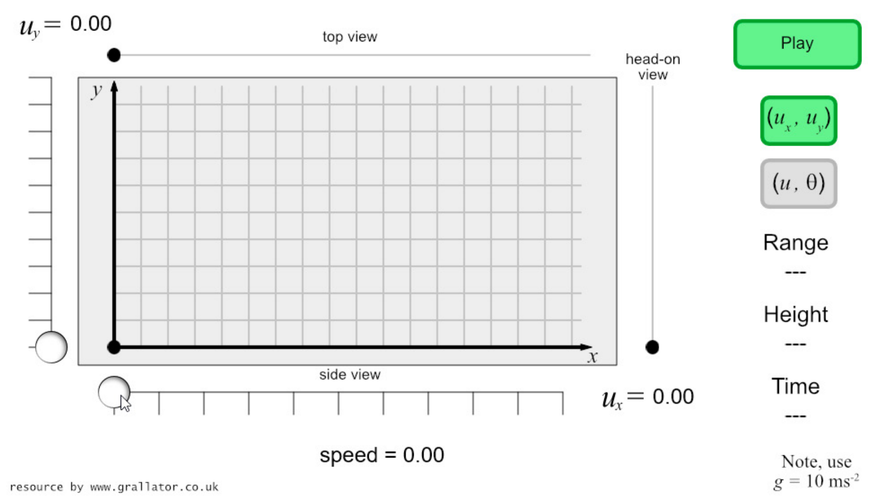

Figure 2 Resource screenshot

1) Set the value of u_x to zero and vary the value of uy as 2, 4, 8, 16 and 32 ms^{-1} to fill in the table below. What do you notice about the height reached and the time the projectile is in the air each time the value of u_y is doubled?

u_y (ms^{-1})	2	4	8	16	32
Height (m)					
Time (s)					

Table 1

2) Set the value of u_y to 40 ms^{-1} and vary the value of u_x as 2, 4, 8 and 16 to fill in the table below. What do you notice about the range, the height reached and the time the projectile is in the air each time the value of u_x is doubled? Explain the result.

u_x (ms^{-1})	2	4	8	16
Range (m)				
Height (m)				
Time (s)				

Table 2

3) Use the SUVAT equation $v^2 = u^2 + 2as$ to predict the maximum height of a project that has an initial vertical velocity of 30 ms^{-1} upwards and an initial horizontal velocity of 10 ms^{-1} to the right. Use g = 10 ms^{-2} vertically downwards. *Hint: think about the value of v you should use.*

4) Use the SUVAT equation $s_y = u_y t - \frac{1}{2} gt^2$ to find the time the projectile is in the air. Explain why this equation gives two values. Use g = 10 ms^{-2}. *Hint: think about the value of s_y you should use.*

5) Use the SUVAT equation $s_x = u_x t$ and the result from 4 to calculate the range of the projectile.

6) Check your answers to 3, 4 and 5 with the resource **SUVATprojectile1.html**

Background – speed and angle

In Activity 2 you experimented with independently changing the horizontal and vertical components of initial velocity. In many cases you do not have the freedom to do this, instead you can change the speed and angle at which the particle is launched, for example for water pumped from a nozzle you can turn the pump flow rate up or down and you can point the nozzle up or down. Given the speed and angle, you have to use trigonometry to find the horizontal and vertical components (this is called resolving the components), as shown in **Figure 3**.

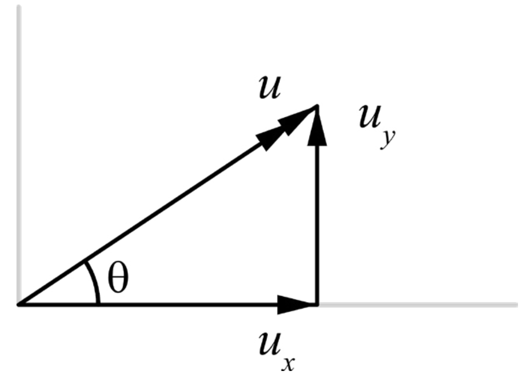

Figure 3
Components of initial velocity

The horizontal and vertical components of speed u at an angle θ relative to the horizontal are given as

$$u_x = u \cos \theta$$

$$u_y = u \sin \theta$$

and, using Pythagoras' theorem

$$u^2 = u_x^2 + u_y^2$$

The last equation is used in the resource **SUVATprojectile1.html** to calculate the launch speed when the values of u_x and u_y are given or changed.

Activity 3 – Launch speed and angle

Selecting the (u, θ) option in the resource **SUVATprojectile1.html** now specifies a speed and angle of launch, as shown in **Figure 4**.

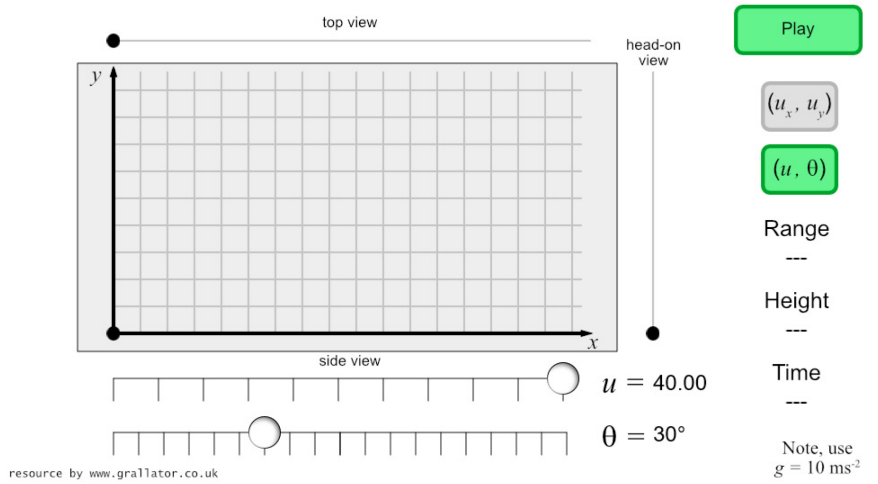

Figure 4 Screen shot of resource in the (u, q) configuration

1) Set the initial speed to 40 ms⁻¹. At what angle does the projectile achieve the maximum possible height? *Note, when θ = 0, the projectile does not move when you click 'Play' as it has already hit the ground, and the simulation stops when this occurs.*

2) Keeping the initial speed at 40 ms⁻¹, find the angle at which the maximum range is achieved. Does the angle that gives the maximum range for a given initial speed depend on the initial speed?

3) What is the relationship between the maximum possible height the projectile can achieve (found in one) and the maximum possible range the projectile can achieve (found in two)?

4) A fireboat has a water nozzle speed of 40 ms⁻¹. Use the resource **SUVATprojectile1.html** to find the angle that the nozzle should be pointed relative to the horizontal in order to reach a fire on another boat that is 80 metres away? Is the answer unique? *Note, the grid lines are 10 metres apart.*

5) A fireboat has a water nozzle speed of 40 ms⁻¹ and is pointed 60 degrees relative to the horizontal. Use the resource **SUVATprojectile1.html** to estimate how far away from a bridge fire that is 40 metres above the level of the nozzle. *Note, the grid lines are 10 metres apart.*

Stretch and challenge activity 1 – mathematics

In Activity 3, part 4 you found the angle or angles (15° and 75°) that the fireboat must angle the nozzle with a nozzle speed of 40 ms⁻¹ to reach a fire on another boat that is 80 metres away. Use the SUVAT equations $s_x = u_x t$ and $s_y = u_y t - \frac{1}{2} g t^2$ to verify the answer. Use g = 10 ms⁻².

You will need to use the angle identity $\sin\theta\cos\theta = \frac{1}{2}\sin 2\theta$

Stretch and challenge activity 2 – engineering

A fireboat has a water nozzle diameter of 20 centimetres. The operational specifications are that the water jet has a maximum range of at least 122.5 metres and a maximum vertical spray height of at least 61.25 metres. Calculate the minimum flow rate the pump must be capable of supplying in litres per second (ls⁻¹) to meet the specifications. Give your answer to the nearest whole number.

Hint: The flow rate in m³s⁻¹, f, of water passing through a nozzle of area A m² at a speed of v ms⁻¹ is given by f = Av.

Notes and solutions

Activity 1

1. The water leaves the nozzle in a stream and travels upwards and to the right. Gravity acts on the stream to pull it back down towards the water upon which the boat is floating.
2. The path the water stream takes in the idealised image is a parabola, for example the curve is a quadratic.

Background – projectile motion

1. The other assumption made is that air resistance is neglected. For small water streams, for example those coming from a small fountain, the assumption is valid. For larger water streams, air resistance becomes more important as it limits range and tends to break the stream up into a spray. However, the mathematics is a little too complex for this level and the assumption of zero air resistance will be used.

Activity 2

1. The results are shown in **Table 3**.

Table 3

u_y (ms^{-1})	2	4	8	16	32
Height (m)	0.2	0.8	3.2	12.8	51.2
Time (s)	0.4	0.8	1.6	3.2	6.4

When the value of u_y is doubled, the height reached increases by a factor 4. The time of flight doubles when u_y is doubled.

2. The results are shown in **Table 4**.

Table 4

u_x (ms^{-1})	2	4	8	16
Range (m)	16	32	64	128
Height (m)	80	80	80	80
Time (s)	8	8	8	8

When the value of u_x is doubled the range doubles. The height and the time are unaffected by changes in u_x. This happens because the horizontal and vertical components of the motion are independent. The vertical motion always starts with an initial speed of $u_y = 40\ ms^{-1}$ so the height reached, and time taken to reach the height and return is always the same, and is independent of u_x. Horizontally, the range is set by how far an object travelling at speed u_x travels in the given time that it is in flight for. As the time is fixed, a doubling of u_x leads to a doubling of the range.

3 Writing the SUVAT equation $v^2 = u^2 + 2as$ in terms of the vertical components of motion (remember g is vertically downwards and so is negative):

$$v_y^2 = u_y^2 - 2gs_y$$

Rearranging to give s_y:

$$s_y = \frac{u_y^2 - v_y^2}{2g}$$

The maximum height is reached when the particle just stops moving upwards and starts to drop back down. At this point $v_y = 0$. Using this, $u_y = 30$ ms^{-1} and g = 10 ms^{-2}, gives the maximum height as:

$$s_y^{max} = \frac{u_y^2 - 0^2}{2g} = \frac{u_y^2}{2g} = \frac{30^2}{2 \times 10}$$

$$s_y^{max} = \frac{900}{20} = 45 \text{ m}$$

4 The vertical displacement is given by $s_y = u_y t - \frac{1}{2}gt^2$, where $u_y = 30$ ms^{-1} and g = 10 ms^{-2} vertically downwards (hence is negative in the equation). The time the particle is in the air is the time at which $s_y = 0$, for example the particle has travelled upwards and returned to its starting level. The equation is then:

$$0 = 30t - \frac{1}{2} \times 10 \times t^2 = 30t - 5t^2$$

This expression is a quadratic and so has two values of t. Factorising:

$$5t(6 - t) = 0$$

$$t(6 - t) = 0$$

This is zero when t = 0, which corresponds to the particle at the beginning of the motion where it is, by definition, at the starting level. The second value of t at which s_y is zero is given by t = 6 s, which is the time at which the particle has travelled upwards and back down again.

More generally, the non-zero time for which $s_y = 0$ is given by cancelling t in the expression $0 = u_y t - \frac{1}{2}gt^2$, so that:

$$\frac{1}{2}gt = u_y \ (t \neq 0)$$

$$t = \frac{2u_y}{g}$$

$$t = \frac{2 \times 30}{10} = 6$$

5) It is stated that $u_x = 10\ \text{ms}^{-1}$. From the previous result, the particle will travel for 6 s. The range is given by $s_x = u_x t$ as:

$$s_x = u_x t = 10 \times 6 = 60 \text{ m}$$

More generally, using the result for t in 4 above,the range is given by:

$$s_x = u_x t = \frac{2u_x u_y}{g}$$

Activity 3

1) The maximum possible height of 80 metres is reached when the angle is 90° relative to the horizontal.

2) The maximum range of 160 metres is reached when the angle is 45° relative to the horizontal. The maximum range is always achieved at this angle irrespective of the initial speed (although the maximum range for a lower initial speed is lower).

3) The maximum possible height a projectile can achieve (angle = 90°) is half the maximum possible range a projectile can achieve (angle = 45°).

4) There are two possible angles that give a range of 80 metres: 15° and 75°. This is not a special case and in general the range achieved at an angle θ is the same as the range at an angle $90 - \theta$. Experiment using the resource **SUVATprojectile1.html**.

5) The resource **SUVATprojectile1.html** indicates that there are two positions at which the stream is at a height of 40 metres. These are about 30 metres away (the water hits the bridge on the way up) and about 108 metres away (the water hits the bridge on the way down), see **Figure 5** (green lines added for construction).

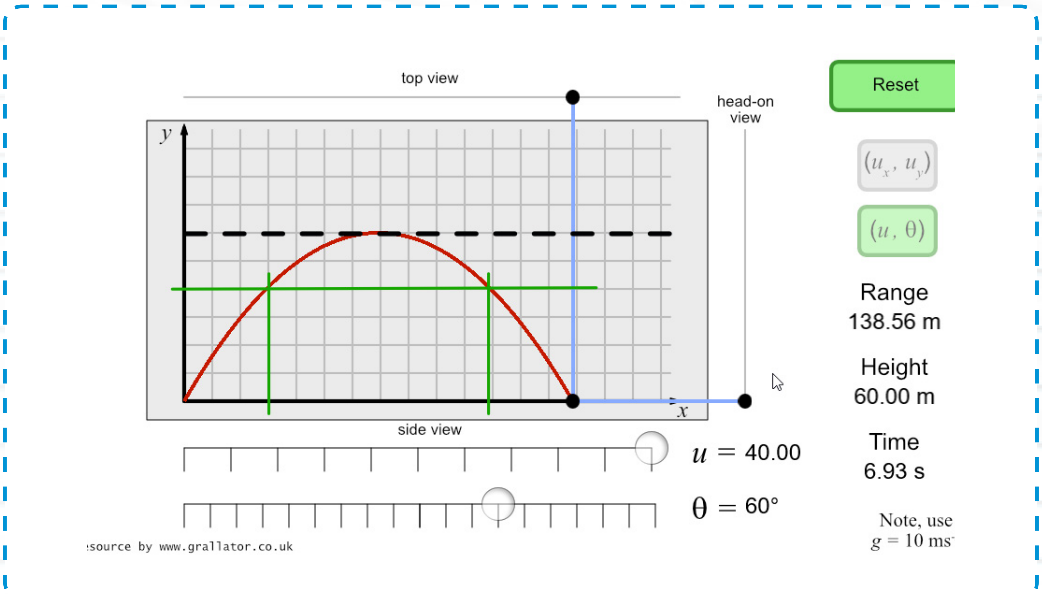

Figure 5
Screen shot of resource estimating two possible ranges

Stretch and challenge activity 1 - mathematics

The problem is shown in the diagram in **Figure 6**.

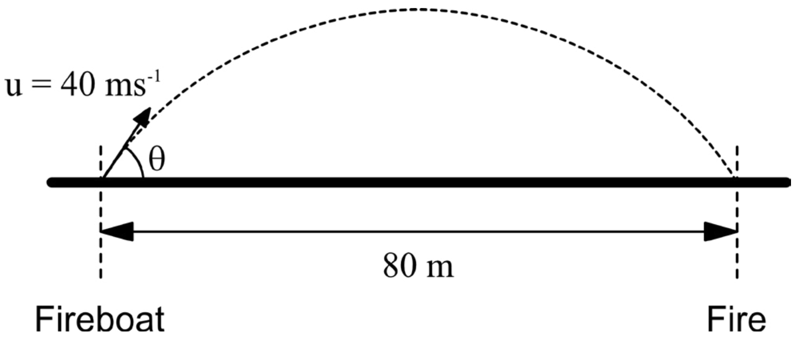

Figure 6
The fireboat and fire problem

The horizontal and vertical SUVAT equations are:

Horizontal: $s_x = u_x t$

Vertical: $s_y = u_y t - \frac{1}{2} g t^2$

where the initial horizontal and vertical components of velocity are:

$u_x = 40 \cos \theta$

$u_y = 40 \sin \theta$

Look first at the vertical component of motion to determine the times at which the vertical displacement is zero (one of these will be t = 0, the starting point, while the other will be time taken to travel upwards, reach a peak height, then fall back down to the level of the water). *Note, this calculation will use $g = 10\ ms^{-2}$.*

$$s_y = u_y t - \frac{1}{2} g t^2$$

$$0 = 40t \sin \theta - \frac{1}{2} \cdot 10t^2$$

$$0 = 40t \sin \theta - 5t^2$$

$$0 = 5t\,(8 \sin \theta - t)$$

This has solutions $t = 0$ or $t = 8 \sin \theta$. From the above, we require the non-zero solution.

Substituting $t = 8 \sin \theta$ into the horizontal component equation and noting that the required horizontal displacement is 80 metres.

$$s_x = u_x t$$

$$80 = 40 \cos \theta \times 8 \sin \theta$$

$$80 = 320 \sin \theta \cos \theta$$

$$\sin \theta \cos \theta = \frac{80}{320} = \frac{1}{4}$$

Using the angle identity $\sin\theta\cos\theta = \frac{1}{2}\sin 2\theta$

$$\frac{1}{2}\sin 2\theta = \frac{1}{4}$$

$$\sin 2\theta = \frac{1}{2}$$

So that

$$2\theta = \sin^{-1}\left(\frac{1}{2}\right)$$

The last expression has two values for which it is satisfied: $\sin 30° = \frac{1}{2}$ and $\sin 150° = \frac{1}{2}$, so that

$$2\theta = 30° \Rightarrow \theta = 15°$$

or

$$2\theta = 150° \Rightarrow \theta = 75°$$

These values agree with those found using the resource **SUVATprojectile1.html**.

Stretch and challenge activity 2 - engineering

The maximum range required is 122.5 metres. Using this with $s_x^{max} = \frac{u^2}{g}$ (g = 10 ms^{-2}) gives the required initial speed of:

$$122.5 = \frac{u^2}{10} \Rightarrow u = \sqrt{1225} = 35 \text{ ms}^{-1}$$

Alternatively, using the maximum achievable height of 61.25 m in $s_y^{max} = \frac{u^2\sin^2\theta}{2g}$, when θ = 90° gives:

$$61.25 = \frac{u^2\sin^2 90}{2\times 10} \Rightarrow u = \sqrt{\frac{1225}{1^2}} = 35 \text{ ms}^{-1}$$

The area of the nozzle is:

$$A = \pi r^2 = \pi \times \left(\frac{20}{2}\right)^2 = 100\pi \text{ cm}^2 = 0.01\pi \text{ m}^2$$

The flow rate, f, is given by $f = Av$ as:

$$f = 0.01\pi \times 35 = 1.10 \text{ m}^3\text{s}^{-1} \quad (2 \text{ d.p.})$$

Converting to ls^{-1} by multiplying by 1000 gives (keeping digits in the calculator):

$$f = 1099.56 \text{ ls}^{-1} \quad (2 \text{ d.p.})$$

The pump flow rate should be at least 1100 ls^{-1}, rounded to the nearest value.