Motion of a tube train – SUVAT

Mechanical engineering

- ✓ Displacement
- ✓ Speed
- ✓ Velocity
- ✓ Acceleration
- ✓ SUVAT equations
- ✓ Distance-time graphs
- ✓ Velocity-time graphs
- ✓ Acceleration-time graphs

Mathematics

- ✓ Algebra
- ✓ Graphs
- ✓ Gradients
- ✓ Quadratic equations

Prerequisites

None

Problem statement

Transport systems such as the London Underground move an enormous number of people in comfort and safety between a large number of stations.

To achieve this, a knowledge of where trains are and how fast they are moving is vital to ensure that a reliable timetable is kept, and that trains are spread safely throughout the network. How can the equations of motion be used to plan such a transport system?

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Activity 1 – Discussion

How and why are the following quantities important for planning a train journey between stations on a network? What restrictions might be imposed on them?

1. The distance between the stations
2. The speed of the train
3. The acceleration of the train away from a station
4. The deceleration of the train into a station
5. The physical location of the underground stations

Background – the SUVAT equations

The SUVAT equations are so-called because they refer to the important quantities:

- displacement (s) **S**
- initial velocity (u) **U**
- final velocity (v) **V**
- uniform acceleration (a) **A**
- time (t) **T**

In the context used here, the displacement can be considered as the distance travelled. Beware, however, this correctly refers to the distance from the starting point of motion to the finishing point of motion, and not the distance travelled; these are not necessarily the same thing! For example, if you throw a ball up in the air and catch it when it returns, it will have travelled a non-zero distance up and then down again. However, the displacement at the end of the motion will be zero as the ball came back to the starting point.

Speed is how fast something is travelling. This is a scalar quantity, for example it does not tell you in which direction an object is travelling, just how fast it is travelling. For SUVAT equations the direction of travel is important, therefore velocities are used. A convention is usually taken such that a positive velocity is to the right while a negative velocity is to the left. For this context, the motion is always in one direction so speed and velocity can be interchanged.

Acceleration is the rate of change of velocity. As velocity is a vector quantity, then acceleration must also be a vector quantity. In the convention that positive velocity is to the right, a positive acceleration will make the velocity more positive, which means in terms of speed:

- if an object is moving to the right its speed will increase
- if an object is moving to the left its speed will decrease (it may also stop then start moving to the right is the acceleration is applied for long enough).

Similarly, a negative acceleration will make the velocity more negative, which means in terms of speed:

- if an object is moving to the right its speed will decrease (it may also stop then start moving to the left is the acceleration is applied for long enough)
- if an object is moving to the left its speed will increase.

For this context, the motion is always in one direction so that when velocity is increasing the acceleration is positive and when velocity is decreasing the acceleration is negative. A deceleration is therefore a negative acceleration.

The SUVAT equations when acceleration is uniform are:

$$s = \frac{1}{2}(u + v)t$$

$$s = ut + \frac{1}{2}at^2$$

$$v = u + at$$

$$v^2 = u^2 + 2as$$

Activity 2 - The three stages of a train journey

A train leaves a station and uniformly accelerates up to its maximum permitted running speed, which it then maintains until it approaches the next station, at which point it uniformly decelerates in order to come to rest at the platform.

Figure 1
Axes for a velocity-time graph

1. Sketch a graph of how the speed of the train varies with time during this journey on axes similar to those shown in **Figure 1**.
   Hint: Break the journey down into three stages.
2. Sketch a graph of how the acceleration varies with time during this journey.
   Hint: The acceleration is constant in each of the three stages of the journey. Think about whether it is positive, negative or zero in each of these stages.
3. Qualitatively, what are the appropriate values for v, u and a in the SUVAT equation $v = u + at$ for the three stages of the journey?
   Hint: Are the values positive, negative or zero? How are the related to the maximum allowed speed?
4. Harder: sketch a graph of how the distance travelled varies with time.
   Hint: the displacement equation, $s = ut + \frac{1}{2}at^2$ gives a clue as to the shape for each of the three stages.

The resource **SUVATunderground.html** can be used to depict the stages of a journey on a simplified underground system consisting of four stations served by a looped track. The quantities s, v and a can be shown individually or simultaneously by using the appropriate selector buttons. Additionally, if the 'Pause at key points' option is selected the journey will be broken down into the acceleration stage, the constant speed stage and the deceleration stage.

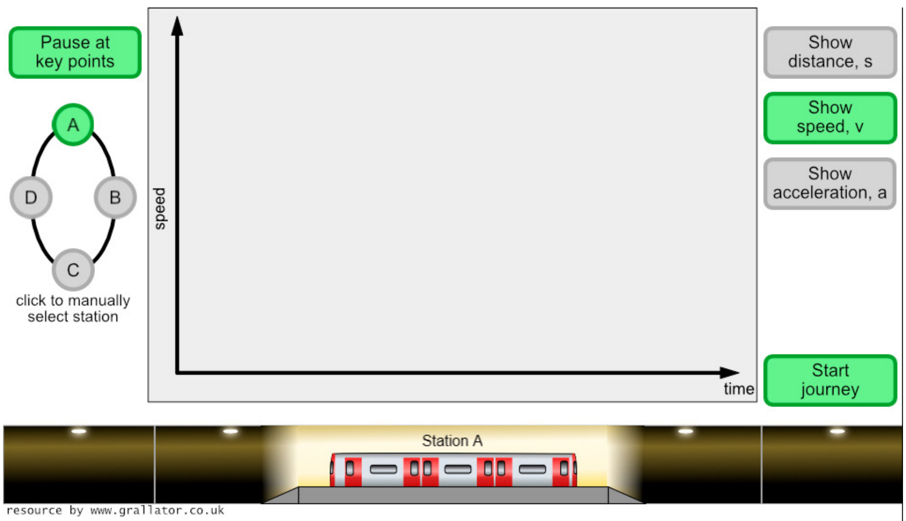

Figure 2
Screen shot of resource

Activity 3 – Analysing the velocity-time graph

The 'Journey statistics' button on the resource **SUVATunderground.html** will show details of the journey just made on screen. Start from 'Station A' (if you are not currently at Station A, click the 'A' in the green circle on the left of the screen to select it as your starting point).

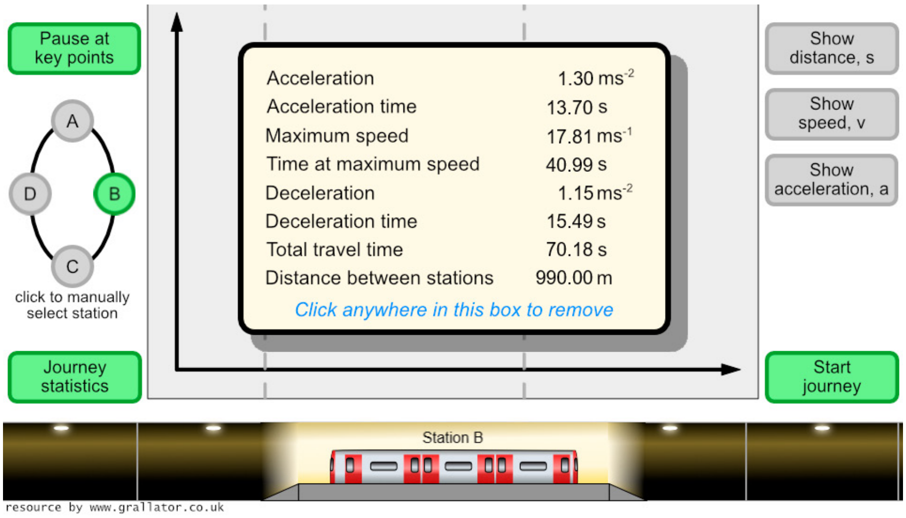

Figure 3
Screen shot of resource showing journey statistics

For the journey Station A to Station B, the statistics presented (to 2 d.p.) are shown in **Table 1**.

Table 1
Journey statistics when travelling from Station A to Station B

Acceleration	1.30	ms^{-2}
Acceleration time	13.70	s
Maximum speed	17.81	ms^{-1}
Time at maximum speed	40.99	s
Deceleration	1.15	ms^{-2}
Deceleration time	15.49	s
Total travel time	70.18	s
Distance between stations	990.00	m

1. Add appropriate values to your sketch of the velocity-time graph.
2. Find the gradient of the line in each of the three different stages of the journey. What connections exist between the values you calculate and the values given in **Table 1**?
3. Find the area under the graph. What connection is there between the calculated value and the values given in **Table 1**?

Activity 4 – SUVAT analysis

Use the SUVAT equations to verify:

1. the acceleration time required to reach the maximum speed allowed for this section of track
2. the deceleration time required to stop at the next station.

The section of track between Station A and Station B is to be marked to give the driver a visual guide as to when to stop accelerating out of Station A, and when to start braking for the approach to Station B.

3. How far out of Station A should the marker to stop accelerating be placed?
4. How far before Station B should the marker to begin breaking be placed?
5. When performing engineering calculations, it is a good habit to have a method of checking calculation values. Verify the calculations for 3 and 4 are consistent by using the values in **Table 1** to calculate the distance travelled at a constant speed and showing the sum of these three distances is equal to the distances between the stations.

Extension/variation – look at other sections of the track

Stretch and challenge activity

A track and train update has been made to the system. For the section between Station A and Station B the train now has the following restrictions:

Maximum acceleration:	1.30 ms^{-2}
Maximum speed:	22.44 ms^{-1}
Maximum deceleration:	1.25 ms^{-2}

1. Calculate the new journey statistics for this section to replace values shown in **Table 1** and use the results to state the new positions for the marker that indicate the driver should stop accelerating and the marker that indicates the driver should start braking.

Notes and solutions

Activity 1

1. The distance between the stations will determine how long a journey at a particular speed will take. Tunnels can be engineered to be very long so any limitations on a distance between stations generally arises from other considerations such as the economics of building the tunnel and safety factors that must be taken into account in, for example, the case of an accident or emergency.

2. Similar to 1. the speed of the train will determine how long a journey between two stations will take. The speed may be limited by several factors including:
   a. the maximum power available to the train
   b. safety and reliability
   c. the need to avoid other trains on the network.

3. The acceleration of the train away from a station determines how quickly the train gains speed from a standing start towards its eventual maximum speed. High accelerations are undesirable from a passenger comfort and safety perspective; it is not safe practice to cause standing passengers to fall over. (A typical value for a comfortable acceleration is 1.3 ms^{-2}. Acceleration will also be dependent on the power available to the train.)

4. The deceleration of the train into a station determines how quickly the train loses speed to come to a halt. As with acceleration, a high deceleration is undesirable from a passenger comfort and safety point of view. Additionally, it may be determined by the braking system used. In a traditional system that uses friction pads to slow the train, heat and brake dust will be generated that needs to be removed from the tunnel system. Harder braking increases the rate of production and requires more frequent parts replacement. Newer systems use regenerative braking in which the energy of braking is stored and used again in the acceleration phase. (Typical decelerations are 1.15 ms^{-2} to 1.3 ms^{-2}.)

5. At the most basic level, the physical location of the underground station determines the distance between stations. However, there is further complexity. Underground tunnels must avoid each other and other engineering systems such as water, gas and electricity supplies and the station must be placed to avoid these. This can mean that stations are at different depths, so there may be a gradient to ascend out of a station, which will affect acceleration, or a gradient to descend into a station, which will affect the deceleration. Further, it may be that stations cannot be connected by straight tracks; curved sections may be required. There will be a restriction on speed through a curved section to maintain comfort to passengers and to maintain train stability on the track.

Activity 2

1. The resource **SUVATunderground.html** gives the following plot for the velocity-time graph.

Figure 4
The velocity-time graph

2. The resource **SUVATunderground.html** gives the following plot for the acceleration-time graph (shown with the velocity-time graph for reference).

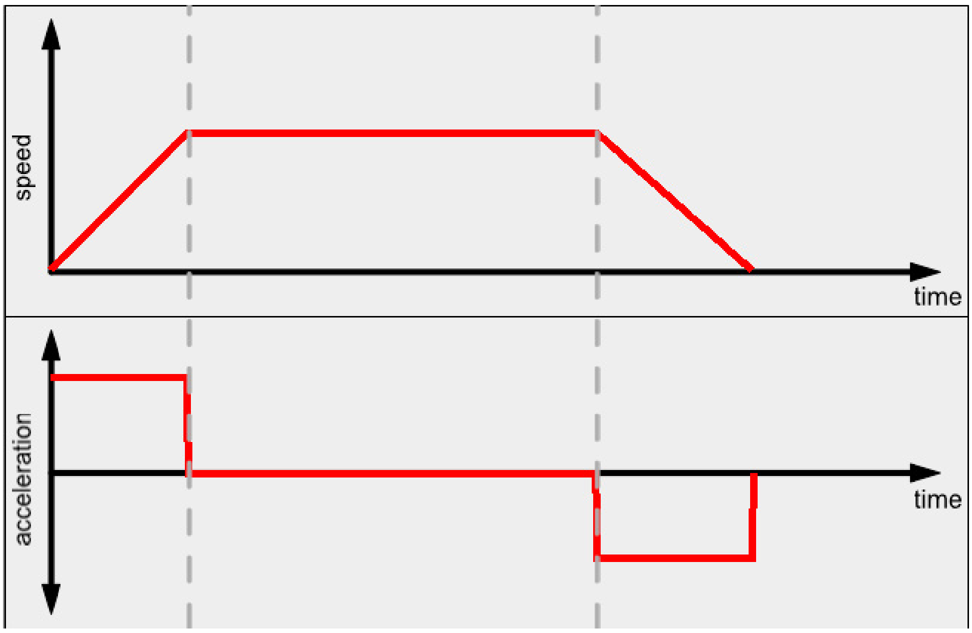

Figure 5
The acceleration-time graph shown with the velocity-time graph

When the velocity is increasing the acceleration must be positive. When the velocity is constant there is no acceleration. When the velocity is reducing the acceleration is negative – a deceleration is therefore a negative acceleration.

3 In the first part of the journey (the accelerating phase), the train goes from being stationary to moving at the maximum allowed speed. This means that:

- u = 0
- v = maximum allowed speed
- a = maximum allowed acceleration

For the second part of the journey the train is moving at constant speed, so:

- u = maximum allowed speed
- v = maximum allowed speed
- a = 0

For the third part of the journey (the decelerating or braking phase), the train goes from moving at the maximum allowed speed to being stationary:

- u = maximum allowed speed
- v = 0
- a = maximum allowed deceleration (negative acceleration)

4 The resource **SUVATunderground.html** gives the following plot for the acceleration-time graph (shown with the velocity-time graph for reference).

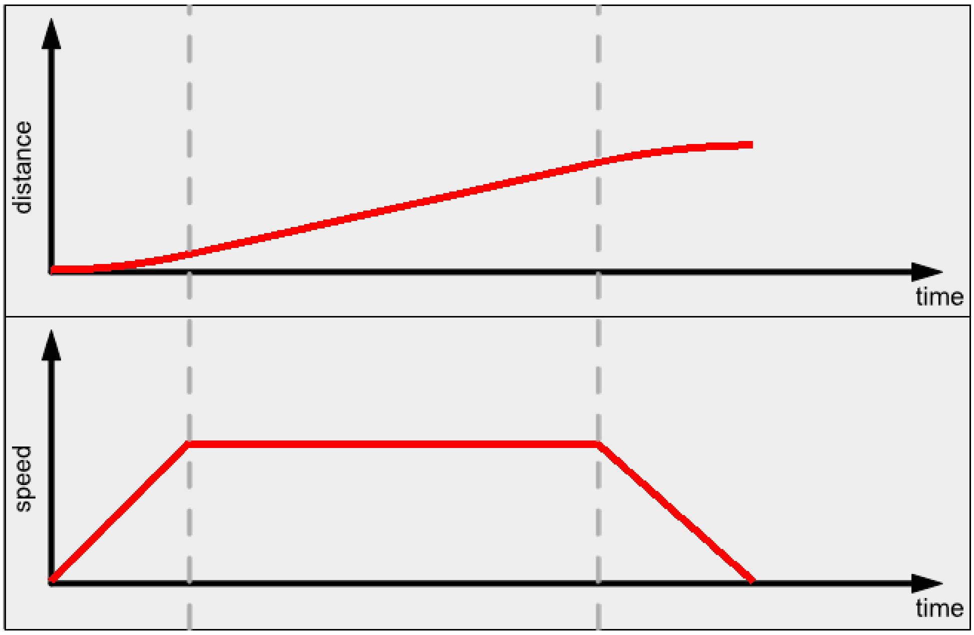

Figure 6
The distance-time graph shown with the velocity-time graph

The distance travelled is related to time by the SUVAT equation $s = ut + \frac{1}{2}at^2$. For the first phase, a > 0 and u = 0, so the shape of the graph is the quadratic $s = \frac{1}{2}at^2$. Physically this makes sense as when the object is accelerating it is getting faster, and as it gets faster it travels a greater and greater distance each second.

For the second phase, a = 0 and u = maximum allowed speed, so the shape of the graph is the straight line $s = ut$.

For the third phase a < 0 and u = maximum allowed speed, so the shape of the graph is the quadratic $s = ut - \frac{1}{2}at^2$ (recall a deceleration is a negative acceleration).

The shape of the graph is clearer when just the distance is plotted in the resource **SUVATunderground.html**.

Figure 7
The distance-time graph shown with the velocity-time graph

Activity 3

1. The speed and time values are shown in **Figure X**. Note, the journey statistics give the duration of each of the three phases, they must be summed to obtain the correct time values from the start of the journey.

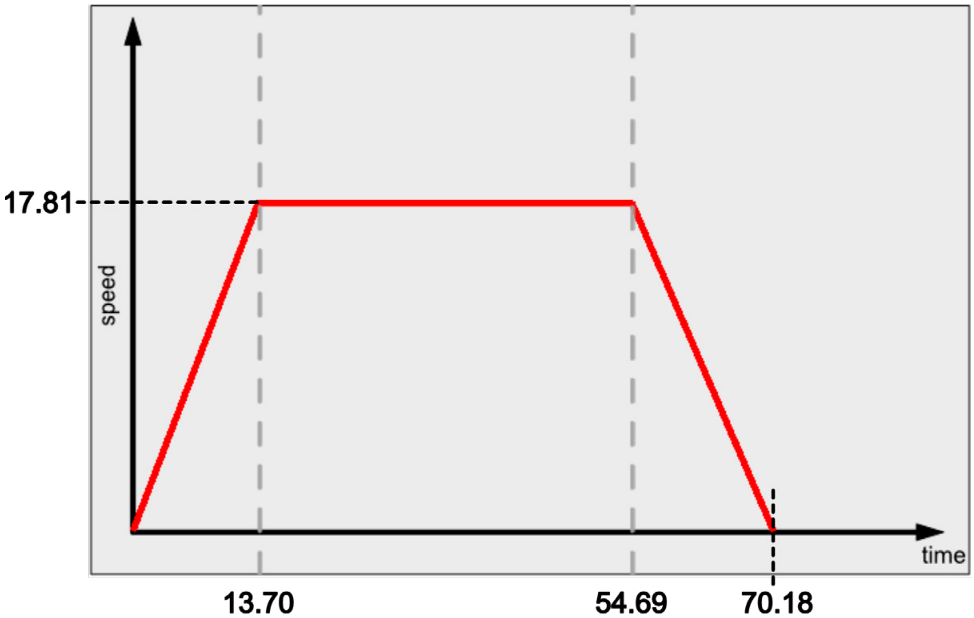

Figure 8
The velocity-time graph with values

2 The gradient is given by $m = \dfrac{\textit{Change in y value}}{\textit{Change in x value}}$.

Phase 1: $m = \dfrac{17.81 - 0}{13.7 - 0} = \dfrac{17.81}{13.7} = 1.3$

Phase 2: $m = \dfrac{17.81 - 17.81}{54.69 - 13.7} = \dfrac{0}{40.99} = 0$

Phase 3: $m = \dfrac{0 - 17.81}{70.18 - 54.69} = \dfrac{-17.81}{54.69} = -1.15\ (2\ d.p.)$

The gradient of the line in each of the three phases is each to the acceleration during that phase (recall a deceleration is a negative acceleration).

3 The area under the graph can be found by summing the areas for the three phases (triangle + rectangle + triangle):

Phase 1 area: $A_1 = \frac{1}{2} \times 13.7 \times 17.81 = 121.9985$

Phase 2 area: $A_2 = 40.99 \times 17.81 = 730.0319$

Phase 3 area: $A_3 = \frac{1}{2} \times 15.49 \times 17.81 = 137.93845$

Summing the areas gives a total area of 989.97. This is very close to the distance travelled: 990 metres. The values given in the journey statistics are rounded to 2 d.p. Had the full number of decimal places been available the areas would sum exactly to 990 metres.

This is an important result: **the area under a velocity-time graph is equal to the displacement, s**.

Activity 4

1 Using $v = u + at$ with v = 17.81, u = 0, a = 1.3 gives:

$$v = u + at \Longrightarrow t = \frac{v - u}{a}$$

$$t = \frac{17.81 - 0}{1.3} = 13.7\ \text{s}$$

2 Using $v = u + at$ with v = 0, u = 17.81, a = −1.15 gives:

$$v = u + at \Longrightarrow t = \frac{v - u}{a}$$

$$t = \frac{0 - 17.81}{-1.15} = 15.49\ \text{s (2 d.p.)}$$

The section of track between Station A and Station B is to be marked to give the driver a visual guide as to when to stop accelerating out of Station A, and when to start braking for the approach to Station B.

$$s = \frac{1}{2}(u + v)t$$

$$s = ut + \frac{1}{2}at^2$$

$$v = u + at$$

$$v^2 = u^2 + 2as$$

3 The distance travelled to the point where the marker to stop accelerating should be placed can be calculated in two ways:

Method 1: Use $s = ut + \frac{1}{2}at^2$.

The known values are u = 0, a = 1.3 and t = 13.7. Substituting gives:

$$s = 0 \times 13.7 + \frac{1}{2} \times 1.3 \times 13.7^2 = 122.00 \text{ m (2 d.p.)}$$

Method 2: Use $v^2 = u^2 + 2as$

The known values are u = 0, v = 17.81 and a = 1.3. Substituting gives:

$$v^2 = u^2 + 2as \Longrightarrow s = \frac{v^2 - u^2}{2a}$$

$$s = \frac{17.81^2 - 0^2}{2 \times 1.3} = 122.00 \text{ m (2 d.p.)}$$

The marker should be 122.00 metres from the station.

4 As with 3 above, the distance before the station can be calculated in two ways:

Method 1: Use $s = ut + \frac{1}{2}at^2$.

The known values are u = 17.81, a = –1.15 and t = 15.49. Note, 15.49 is a value rounded to 2 d.p., for this calculation you should use the value to the full number of decimal places, calculated in 2 above. Substituting gives:

$$s = 17.81 \times 15.49 - \frac{1}{2} \times 1.15 \times 15.49^2 = 137.91 \text{ m (2 d.p.)}$$

Method 2: Use $v^2 = u^2 + 2as$.

The known values are u = 17.81, v = 0 and a = –1.15. Substituting gives:

$$v^2 = u^2 + 2as \Longrightarrow s = \frac{v^2 - u^2}{2a}$$

$$s = \frac{0^2 - (17.81)^2}{2 \times -1.15} = 137.91 \text{ m (2 d.p.)}$$

The marker should be 37.91 metres before the next station.

5 Distance travelled in 40.99 s when travelling at a constant speed of 17.81 ms^{-1} (a = 0) is given by:

$$s = ut + \frac{1}{2}at^2$$

$$s = 17.81 \times 40.99 + \frac{1}{2} \times 0 \times 40.99^2$$

$$s = 730.03 \text{ m (2 d.p.)}$$

Summing the distance calculated in 3 (122 metres) and 4 (137.91 metres), and this value (730.03 metres) gives a total distance of 990 metres, as expected.

Stretch and challenge activity

1 The shaded lines of **Table 2** need updating by calculation. The non-shaded values are given, and the shaded ones are to be calculated.

Acceleration	1.30	ms^{-2}
Acceleration time	13.70	s
Maximum speed	22.44	ms^{-1}
Time at maximum speed	40.99	s
Deceleration	1.25	ms^{-2}
Deceleration time	15.49	s
Total travel time	70.18	s
Distance between stations	990.00	m

Table 2
Updated values and shaded values need changing for journey statistics when travelling from Station A to Station B

Acceleration time. Using $v = u + at$ with v = 22.44, u = 0, a = 1.3 gives

$$v = u + at \Longrightarrow t = \frac{v - u}{a}$$

$$t = \frac{22.44 - 0}{1.3} = 17.26 \text{ s (2 d.p.)}$$

Deceleration time. Using $v = u + at$ with v = 0, u = 22.44, a = −1.25 gives

$$v = u + at \Longrightarrow t = \frac{v - u}{a}$$

$$t = \frac{0 - 22.44}{-1.25} = 17.95 \text{ s (2 d.p.)}$$

To calculate the time for which the train travels at a constant speed you first need to determine how far it has to travel at a constant speed. To obtain this, you need to calculate how far the train moves in the acceleration and deceleration phases and subtract this from the total distance between the stations. As with Activity 4.3. and 4.4. there are two methods of calculating these values.

Note, retain previous values in the calculator for these calculations.

Acceleration phase distance, method 1: Use $s = ut + \frac{1}{2}at^2$.

The known values are u = 0, a = 1.3 and t = 17.26. Substituting gives:

$$s = 0 \times 17.26 + \frac{1}{2} \times 1.3 \times 17.26^2 = 193.67 \text{ m (2 d.p.)}$$

Acceleration phase distance, method 2: Use $v^2 = u^2 + 2as$.

The known values are u = 0, v = 22.44 and a = 1.3. Substituting gives:

$$v^2 = u^2 + 2as \Longrightarrow s = \frac{v^2 - u^2}{2a}$$

$$s = \frac{22.44^2 - 0^2}{2 \times 1.3} = 193.67 \text{ m (2 d.p.)}$$

Deceleration phase distance, method 1: Use $s = ut + \frac{1}{2}at^2$.

The known values are u = 22.44, a = -1.25 and t = 17.95. Substituting gives:

$$s = 22.44 \times 17.95 - \frac{1}{2} \times 1.25 \times 17.95^2 = 201.42 \text{ m (2 d.p.)}$$

Deceleration phase distance, method 2: Use $v^2 = u^2 + 2as$.

The known values are u = 22.44, v = 0 and a = −1.25. Substituting gives:

$$v^2 = u^2 + 2as \Rightarrow s = \frac{v^2 - u^2}{2a}$$

$$s = \frac{0^2 - (22.44)^2}{2 \times -1.25} = 201.42 \text{ m (2 d.p.)}$$

The total distance to travel is 990 metres so the distance for which the train travels at a constant speed is *(note, values are rounded in this expression – use values to their full decimal places)*:

$$s_c = 990 - 193.67 - 201.42 = 594.90 \text{ m (2 d.p.)}$$

The train travels at a constant speed of 22.44 ms^{-1}. The time taken to cover this distance is given by:

$$t_c = \frac{s_c}{v} = \frac{594.90}{22.44} = 26.51 \text{ s}$$

The total travel time is given by the sum of the times for the individual phases of the journey (again keep all digits from previous calculations)

$$t_{total} = 17.26 + 17.95 + 26.51 = 61.72 \text{ s}$$

The updated table is as shown in **Table 3**.

Acceleration	1.30	ms^{-2}
Acceleration time	17.26	s
Maximum speed	22.44	ms^{-1}
Time at maximum speed	26.51	s
Deceleration	1.25	ms^{-2}
Deceleration time	17.95	s
Total travel time	61.72	s
Distance between stations	990.00	m

Table 3
Updated journey statistics when travelling from Station A to Station B

The acceleration and deceleration distances calculated above show where the markers should be placed on the track.

As an extension exercise, you could work out the area under the velocity-time graph for this journey to show that when the full number of decimal places are kept, the area equals the distance travelled between the stations.

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